Timoshenko beam stiffness calculator

Sometimes blade flexures aren't slender enough to be fully characterized well enough by Euler-Bernoulli beam deflections. This calculator is meant to aid in stiffness calculations for second-order modeling. I'm looking for something practical yet useful (something I can parameterize and throw into a stiffness or compliance matrix).

Based on formulas and derivations given in:


My application is for blade flexures, so I'm focusing on the clamped-clamped case since I can split it in half to get my blade flexure

**Summary**

I've written out here a symbolic correction factor for shear beams that depends only on the depth/length ratio and can be applied as a correction factor to the Euler-Bernoulli deflection, making it easy to plug into a compliance matrix for more accurate stiffness modeling.

\[
k = \frac{12EI}{L^3} \left[ \frac{1}{1 + 3.2 \left( \frac{h}{L} \right)^2} \right]
\]

h/L is no smaller than 0.1 (starts to behave as a wire, not a blade), can be larger (wide sheet flexures)

\[ h/l = 0.5 \text{ gives } 55\% \text{ of Euler stiffness} \]
\[ h/l = 1 \text{ gives } 24\% \]
\[ h/l = 2 \text{ gives } 7\% \]

**Parameters**

- \( b := 0.5\text{-mm} \) \quad \text{Beam depth}
- \( b := b \)
- \( h := 12\text{-mm} \) \quad \text{Beam height}
- \( h := h \)
- \( L_{\text{act}} := 25.4\text{-mm} \)
- \( L := L_{\text{act}} \)
- \( L := 2 L_{\text{act}} \) \quad \text{Effective Beam length}
- \( E := 70\text{-GPa} \) \quad \text{Youngs Modulus}
- \( E := E \)
- \( G := \frac{3}{8} E = 26.25\text{ GPa} \) \quad \text{Shear modulus}
- \( G := G \)

Why do I have “double definitions”? This lets me do both symbolic and numerical computations in the same worksheet. Very useful!
Geometry info

\[ A_{\text{w}} := b \cdot h = 6 \text{ mm}^2 \quad \text{Cross sectional area} \]

\[ A := A \]

\[ I := \frac{1}{12} \cdot b \cdot h^3 = 72 \text{ mm}^4 \quad \text{Area moment of inertia} \]

\[ I := I \]

Euler bernoulli is valid when

\[ \frac{E \cdot I}{\kappa \cdot L^2 \cdot A \cdot G} < 1 \quad \text{must be much less than 1} \]

Deep-beam constants

These constants arise from the solutions of the beam differential equations, which include nasty hyperbolic trig functions. Leave 'em alone.

\[ A_o := \cosh \left( \frac{1}{2} \right) - 12 \cdot \left( \cosh \left( \frac{1}{2} \right) - 2 \cdot \sinh \left( \frac{1}{2} \right) \right) = 0.102 \]

\[ B_o := \cosh \left( \frac{1}{2} \right)^2 + 6 \cdot (\sinh(1) - 1) - 24 \cdot \cosh \left( \frac{1}{2} \right) \cdot \left( \cosh \left( \frac{1}{2} \right) - 2 \cdot \sinh \left( \frac{1}{2} \right) \right) = 0.011 \]

\[ C_o := \cosh \left( \frac{1}{2} \right)^2 + \left( \frac{1}{2} \right) \cdot \left( (\sinh(1) + 1) - 4 \cdot \cosh \left( \frac{1}{2} \right) \cdot \sinh \left( \frac{1}{2} \right) \right) = 8.739 \times 10^{-3} \]

\[ \alpha := \frac{B_o}{A_o} - A_o = 1.195 \times 10^{-3} \]

\[ \beta := \frac{G \cdot A \cdot C_o}{E \cdot I \cdot A_o} = 2.667 \times 10^{3} \frac{1}{\text{m}^2} \]

\[ \lambda := \sqrt{\frac{\beta}{\alpha}} = 1.494 \times 10^{3} \frac{1}{\text{m}} \]

\[ \lambda := \lambda \]

Deflection equations

\[ w(x) := \left[ \frac{P \cdot L^3}{48 \cdot E \cdot I} \cdot \left( \frac{\cosh(\lambda \cdot x) - \sinh(\lambda \cdot x) - 1}{\lambda \cdot L} \right) \right] \]

\[ k(x) := \left[ \frac{L^3}{48 \cdot E \cdot I} \cdot \left( \frac{\cosh(\lambda \cdot x) - \sinh(\lambda \cdot x) - 1}{\lambda \cdot L} \right) \right]^{-1} \]

As described in the paper, for practical problems \( \lambda L \) is very large, so \( \sinh(\lambda L) \sim \cosh(\lambda L) \) and the equations can be simplified.
\[ w(x) := \left[ \frac{P L^3}{48E I} \left( 3 \frac{x^2}{L^2} - 4 \frac{x^3}{L^3} \right) + \frac{3 P L}{5 G A} \left( \frac{x}{L} + \frac{-1}{\lambda L} \right) \right] \]

\[ k(x) := \left[ \frac{L^3}{48E I} \left( 3 \frac{x^2}{L^2} - 4 \frac{x^3}{L^3} \right) + \frac{3 L}{5 G A} \left( \frac{x}{L} + \frac{-1}{\lambda L} \right) \right]^{-1} \]

\[ w\left(\frac{L}{2}\right) \rightarrow \frac{L^3 P}{192 E I} - \frac{3 L P \left( \frac{1}{L \lambda} - \frac{1}{2} \right)}{5 A G} \]

\( L * \lambda \) is large so we can neglect it here to simplify:

\[ \frac{L^3 P}{192 E I} - \frac{3 L P \left( \frac{0 - \frac{1}{2}}{2} \right)}{5 A G} \rightarrow \frac{3 L P}{10 A G} + \frac{L^3 P}{192 E I} \]

The first term can be rewritten to look similar to the first term, factor:

\[ \frac{P L^3}{192 E I} \left[ 1 + 12.8 \left( \frac{h}{L} \right)^2 \right] \]

Typical values of \( h/L \) for a blade flexure would be around 0.1-1, but can be higher (probably not lower since it would start to act more as a wire constraint). The shallower the blade, the less relevant the second term becomes. Let's compare the fixed-fixed beam with the shear correction to the beam without:

\[ k := \frac{192 E I}{L^3} \left[ 1 + 12.8 \left( \frac{h}{L} \right)^2 \right]^{-1} = 4306 \frac{N}{mm} \]

\[ k_{\text{euler}} := \frac{192 E I}{L^3} = 7.381 \times 10^3 \frac{N}{mm} \]

\[ \frac{k}{k_{\text{euler}}} = 0.583 \]

Now we need to bring this back to our simple fixed-guided blade flexure instead of a clamped-clamped blade. A clamped-clamped arrangement is equivalent to two fixed-guided beams in parallel with twice the length of each element (that's why \( L_{\text{eff}} = 2L \)), so to get the stiffness of a single blade we divide by two

\[ k_{fg} := \frac{k}{2} = 2153 \frac{N}{mm} \]

Stiffness of a single blade

For the equation above to be useful, we should rewrite it symbolically in terms of \( L_{\text{act}} \), the length of the blade. It should be the same as dividing the stiffness of the fixed-fixed beam in two!
Deflection equation for clamped clamped beam, as written in the paper:

$$\delta_{\text{max}} := \frac{P \cdot L^3}{192 \cdot E \cdot I} \left[ 1 + 9.6 \left( 1 + \mu \right) \frac{h^2}{L^2} \right]$$

$$k := \frac{192 \cdot E \cdot I}{L^3} \left[ 1 + 9.6 \left( 1 + \mu \right) \left( \frac{h}{L} \right)^2 \right]^{-1}$$

They give these equations in terms of \( \mu \), but don't give \( \mu \) factors, that's why I went through and plugged in values above and came up with a 12.8 factor on the square of the aspect ratio.