Timoshenko beam stiffness calculator

Sometimes blade flexures aren't slender enough to be fully characterized well enough by Euler-Bernoulli beam deflections. This calculator is meant to aid in stiffness calculations for second-order modeling. I'm looking for something practical yet useful (something I can paramterize and throw into a stiffness or compliance matrix).

Based on formulas and derivations given in:

> Y. M. Ghugal and R. Sharma. "A refined shear deformation theory for flexure of thick beams." Latin American Journal of Solids and Structures. 8(2011) 183-195

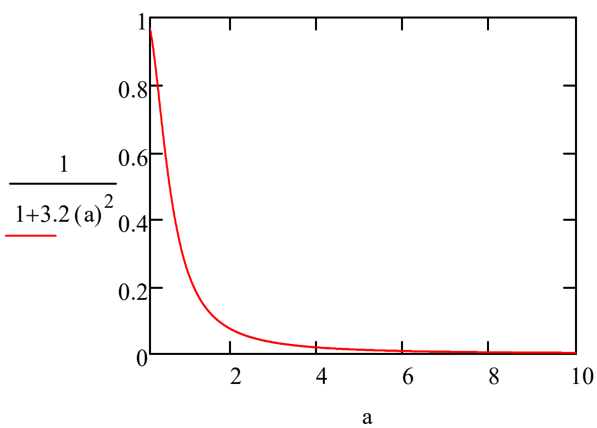
My application is for blade flexures, so I'm focusing on the clamped-clamped case since I can split it in half to get my blade flexure

Summary

I've written out here a symbolic correction factor for shear beams that depends only on the depth/length ratio and can be applied as a correction factor to the Euler-Bernoulli deflection, making it easy to plug into a compliance matrix for more accurate stiffness modeling.

$$k = \frac{12 \cdot E \cdot I}{L^3} \cdot \frac{1}{\left[1 + 3.2 \cdot \left(\frac{h}{L}\right)^2\right]}$$

h/L is no smaller than 0.1 (starts to behave as a wire, not a blade), can be larger (wide sheet flexures)



h/l = 0.5 gives 55% of Euler stiffness
h/l = 1 gives 24%
h/l = 2 gives 7%

Parameters

$b := 0.5 \cdot mm$ Beam depth

$b := b$

$h := 12 \cdot mm$ Beam height

$h := h$

$L_{act} := 25.4 \cdot mm$

$L := L_{act}$

$L := 2 \cdot L_{act}$ Effective Beam length

$L := L$

$E := 70 \cdot GPa$ Youngs Modulus

$E := E$

$G := \frac{3}{8} \cdot E = 26.25\,GPa$ Shear modulus

$G := G$

Why do I have "double definitions"? This lets me do both symbolic and numerical computations in the same worksheet. Very useful!

Geometry info

$A := b \cdot h = 6\,mm^2$ Cross sectional area

$A := A$

$I := \frac{1}{12} \cdot b \cdot h^3 = 72\,mm^4$ Area moment of inertia

$I := I$

Euler bernoulli is valid when

$\frac{E \cdot I}{\kappa \cdot L^2 \cdot A \cdot G} < 1$ must be much less than 1

Deep-beam constants

These constants arise from the solutions of the beam differential equations, which include nasty hyperbolic trig functions. Leave 'em alone.

$$A_o := \cosh\left(\frac{1}{2}\right) - 12 \cdot \left(\cosh\left(\frac{1}{2}\right) - 2 \cdot \sinh\left(\frac{1}{2}\right)\right) = 0.102$$

$$B_o := \cosh\left(\frac{1}{2}\right)^2 + 6 \cdot (\sinh(1) - 1) - 24 \cdot \cosh\left(\frac{1}{2}\right) \cdot \left(\cosh\left(\frac{1}{2}\right) - 2 \cdot \sinh\left(\frac{1}{2}\right)\right) = 0.011$$

$$C_o := \cosh\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) \cdot ((\sinh(1) + 1)) - 4 \cdot \cosh\left(\frac{1}{2}\right) \cdot \sinh\left(\frac{1}{2}\right) = 8.739 \times 10^{-3}$$

$$\alpha := \frac{B_o}{A_o} - A_o = 1.195 \times 10^{-3}$$

$$\beta := \frac{G \cdot A \cdot C_o}{E \cdot I \cdot A_o} = 2.667 \times 10^3 \frac{1}{m^2}$$

$$\lambda := \sqrt{\frac{\beta}{\alpha}} = 1.494 \times 10^3 \frac{1}{m}$$

$\lambda := \lambda$

Deflection equations

$$w(x) := \left[\frac{P \cdot L^3}{48 \cdot E \cdot I} \cdot \left(3 \cdot \frac{x^2}{L^2} - 4 \cdot \frac{x^3}{L^3}\right) + \frac{3}{5} \cdot \frac{P \cdot L}{G \cdot A} \cdot \left(\frac{x}{L} + \frac{\cosh(\lambda \cdot x) - \sinh(\lambda \cdot x) - 1}{\lambda \cdot L}\right)\right]$$

$$k(x) := \left[\frac{L^3}{48 \cdot E \cdot I} \cdot \left(3 \cdot \frac{x^2}{L^2} - 4 \cdot \frac{x^3}{L^3}\right) + \frac{3}{5} \cdot \frac{L}{G \cdot A} \cdot \left(\frac{x}{L} + \frac{\cosh(\lambda \cdot x) - \sinh(\lambda \cdot x) - 1}{\lambda \cdot L}\right)\right]^{-1}$$

As described in the paper, for practical problems λL is very large, so sinh(λL) ~= cosh(λL) and the equations can be simplified.

$$w(x) := \left[\frac{P\cdot L^3}{48\cdot E\cdot I}\cdot\left(3\cdot\frac{x^2}{L^2} - 4\cdot\frac{x^3}{L^3}\right) + \frac{3}{5}\cdot\frac{P\cdot L}{G\cdot A}\cdot\left(\frac{x}{L} + \frac{-1}{\lambda\cdot L}\right)\right]$$

$$k(x) := \left[\frac{L^3}{48\cdot E\cdot I}\cdot\left(3\cdot\frac{x^2}{L^2} - 4\cdot\frac{x^3}{L^3}\right) + \frac{3}{5}\cdot\frac{L}{G\cdot A}\cdot\left(\frac{x}{L} + \frac{-1}{\lambda\cdot L}\right)\right]^{-1}$$

$$w\left(\frac{L}{2}\right) \rightarrow \frac{L^3\cdot P}{192\cdot E\cdot I} - \frac{3\cdot L\cdot P\cdot\left(\frac{1}{L\cdot\lambda} - \frac{1}{2}\right)}{5\cdot A\cdot G}$$

L*λ is large so we can neglect it here to simplify:

$$\frac{L^3\cdot P}{192\cdot E\cdot I} - \frac{3\cdot L\cdot P\cdot\left(0 - \frac{1}{2}\right)}{5\cdot A\cdot G} \rightarrow \frac{3\cdot L\cdot P}{10\cdot A\cdot G} + \frac{L^3\cdot P}{192\cdot E\cdot I}$$

The first term can be rewritten to look similar to the first term, factor:

$$\frac{P\cdot L^3}{192\cdot E\cdot I}\left[1 + 12.8\cdot\left(\frac{h}{L}\right)^2\right]$$

Typical values of h/L for a blade flexure would be around 0.1-1, but can be higher (probably not lower since it would start to act more as a wire constraint) . The shallower the blade, the less relevant the second term becomes. Let's compare the fixed-fixed beam with the shear correction to the beam without:

$$k := \frac{192\cdot E\cdot I}{L^3}\cdot\left[1 + 12.8\cdot\left(\frac{h}{L}\right)^2\right]^{-1} = 4306\,\frac{N}{mm}$$

$$k_{euler} := \frac{192\cdot E\cdot I}{L^3} = 7.381\times 10^3\,\frac{N}{mm}$$

$$\frac{k}{k_{euler}} = 0.583$$

Now we need to bring this back to our simple fixed-guided blade flexure instead of a clamped-clamped blade. A clamped-clamped arrangement is equivalent to two fixed-guided beams in parallel with twice the length of each element (that's why L.eff=2*L), so to get the stiffness of a single blade we divide by two

$$k_{fg} := \frac{k}{2} = 2153\,\frac{N}{mm}$$ Stiffness of a single blade

For the equation above to be useful, we should rewrite it symbolically in terms of L.act, the length of the blade. It should be the same as dividing the stiffness of the fixed-fixed beam in two!

$$k := \frac{12 \cdot E \cdot I}{L_{act}^3} \cdot \left[1 + 3.2 \cdot \left(\frac{h}{L_{act}}\right)^2\right]^{-1} = 2153 \frac{N}{mm}$$

Deflection equation for clamped clamped beam, as written in the paper:

$$\delta_{max} := \frac{P \cdot L^3}{192 \cdot E \cdot I}\left[1 + 9.6 \cdot (1 + \mu) \cdot \frac{h^2}{L^2}\right]$$

$$k := \frac{192 \cdot E \cdot I}{L^3} \cdot \left[1 + 9.6 \cdot (1 + \mu) \cdot \left(\frac{h}{L}\right)^2\right]^{-1}$$

They give these equations in terms of mu, but don't give mu factors, that's why I went through and plugged in values above and came up with a 12.8 factor on the square of the aspect ratio